On the rate of convergence in the central limit theorem for martingale difference sequences

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24 Mars 2004

ABSTRACT: We established the rate of convergence in the central limit theorem for stopped sums of a class of martingale difference sequences.

Sur la vitesse de convergence dans le théorème limite central pour les différences de martingale

RÉSUMÉ: On établit la vitesse de convergence dans le théorème limite central pour les sommes arrêtées issues d'une classe de suites de différences de martingale.

AMS Subject Classifications (2000): 60G42, 60F05
Keywords: central limit theorem, martingale difference sequence, rate of convergence.

1 Introduction

Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of random variables defined on a probability space $(\Omega,\ \mathcal{F},\ \mathbb{P})$. We shall say that $(X_k)_{k\in\mathbb{N}}$ is a martingale difference sequence if, for any $k \geq 0$

1. $\mathbb{E}\{|X_k|\} < +\infty$.
2. $\mathbb{E}\{X_{k+1}|\mathcal{F}_k\} = 0$, where $\mathcal{F}_k$ is the σ-algebra generated by $X_i, i \leq k$.

For each integer $n \geq 1$ and x real number, we denote

$$S_0 = 0, \quad S_n = \sum_{i=1}^{n} X_i, \quad \phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-\frac{t^2}{2})dt, \quad \sigma_{n-1}^2 = \mathbb{E}\{X_n^2|\mathcal{F}_{n-1}\},$$

$$\nu(n) = \inf\{k \in \mathbb{N}^*/\ \sum_{i=0}^{k} \sigma_i^2 \geq n\}, \quad S_{\nu(n)}^2 = \sum_{k=1}^{+\infty} S_k^2\, I_{\nu(n)=k}, \quad \sigma_{\nu(n)}^2 = \sum_{k=1}^{+\infty} \sigma_k^2\, I_{\nu(n)=k},$$

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$$F_n(x) = \mathbb{P}(S_{\nu(n)} \leq x\sqrt{n}), \quad S'_{\nu(n)} = S_{\nu(n)} + \sqrt{\gamma(n)} X_{\nu(n)+1}, \quad H_n(x) = \mathbb{P}(S'_{\nu(n)} \leq x\sqrt{n}),$$

and $\gamma(n)$ is a random variable such that

$$\sum_{i=0}^{\nu(n)-1} \sigma_i^2 + \gamma(n)\sigma_{\nu(n)}^2 = n \qquad p.s. \tag{1}$$

If the random variables X_i are independent and identically distributed with $\mathbb{E}X_i = 0$ and $\mathbb{E}X_i^2 = 1$, we have by the central limit theorem (CLT)

$$\lim_{n\to+\infty} \sup_{x\in\mathbb{R}} |\mathbb{P}(S_n \leq x\sqrt{n}) - \phi(x)| = 0.$$

By the theorem of Berry ([1], 1941) and Esseen ([3], 1942), if moreover, $\mathbb{E}|X_i^3| < +\infty$, the rate of convergence in the limit is of order $n^{-\frac{1}{2}}$. If $(X_i)_{i\in\mathbb{N}}$ is an ergodic martingale difference sequence with $\mathbb{E}X_i^2 = 1$, by the theorem of Billingsley ([9], 1968) and Ibragimov (([6], 1963), see also ([10], 1980)) we have the CLT. The rate of convergence can, however, be arbitrarily slow even if X_i are bounded and α-mixing (cf [7]). There are several results showing that with certain assumption on the conditional variance $\mathbb{E}(X_i^2|\mathcal{F}_{i-1})$, the rate of convergence becomes polynomial (Kato ([13], 1979), Grams ([12], 1972), Nakata ([11], 1976), Bolthausen ([4], 1982), Haeusler ([5], 1988), ...).

In 1963, Ibragimov [6] has shown that for X_i uniformly bounded, if instead of usual sums S_n, the stopped sums $S_{\nu(n)}$ or $S'_{\nu(n)}$ are considered, one gets the rate of convergence of order $n^{-\frac{1}{4}}$; the only assumption beside boundednes is that $\sum_{i=0}^{+\infty} \sigma_i^2$ diverge to infinity a.s.

In the present paper we give a rate of convergence for a larger class of martingale difference sequences, the Ibragimov's case will be a particular one.

2 Main result

We consider a sequence $(X_i)_{i\in\mathbb{N}}$ of square integrable martingale differences.

Theorem 1. *If the series $\sum_{i=0}^{+\infty} \sigma_i^2$ diverges a.s. and if there exists a nondecreasing sequence $(Y_i)_{i\in\mathbb{N}}$ adapted to the filtration $(\mathcal{F}_i,\ i \in \mathbb{N})$ such that, for all $i \in \mathbb{N}^*$*

$$\mathbb{E}(Y_i^4) < +\infty, \quad 1 \leq Y_i \quad \textit{and} \quad \mathbb{E}(|X_i|^3|\mathcal{F}_{i-1}) \leq Y_{i-1}\,\sigma_{i-1}^2 \quad a.s.$$

then for all n sufficiently large

$$\sup_{x\in\mathbb{R}} \Big| F_n(x) - \phi(x) \Big| \leq \frac{a_n^{\frac{1}{2}}}{\pi n^{\frac{1}{4}}} \left(11 + \frac{3}{4n^{\frac{1}{4}}} + \frac{2}{9n^{\frac{1}{2}}} + \frac{1}{8n^{\frac{3}{4}}} \right), \tag{2}$$

$$\sup_{x\in\mathbb{R}} \Big| H_n(x) - \phi(x) \Big| \leq \frac{a_n^{\frac{1}{2}}}{\pi n^{\frac{1}{4}}} \left(11 + \frac{9}{4n^{\frac{1}{4}}} + \frac{2}{9n^{\frac{1}{2}}} + \frac{1}{8n^{\frac{3}{4}}} \right) \tag{3}$$

where $a_n = (\mathbb{E}Y_{\nu(n)}^4)^{\frac{1}{2}}$.

If we put $Y_i = M$ a.s. where $M > 0$ is a constant, one obtains the following corollaries:

Corollary 1. *If the series $\sum_{i=0}^{+\infty} \sigma_i^2$ diverges a.s. and there exists $M > 0$ such that, for all $i \in \mathbb{N}^*$, $\mathbb{E}(|X_i|^3|\mathcal{F}_{i-1}) \leq M\,\mathbb{E}(X_i^2|\mathcal{F}_{i-1})$ a.s. then there is a constant $0 < c_M < +\infty$*

$$\sup_{x\in\mathbb{R}} \Big| F_n(x) - \phi(x) \Big| \leq \frac{c_M}{n^{\frac{1}{4}}}, \tag{4}$$

$$\sup_{x\in\mathbb{R}} \Big| H_n(x) - \phi(x) \Big| \leq \frac{c_M}{n^{\frac{1}{4}}}. \tag{5}$$

Corollary 2. *If there exists $0 < \alpha \leq M < +\infty$ satisfying $\sigma_{i-1}^2 \geq \alpha$ and $\mathbb{E}(|X_i|^3|\mathcal{F}_{i-1}) \leq M$ a.s. for all $i \in \mathbb{N}^*$, then there is a constant $0 < c_{(\alpha,\,M)} < +\infty$ such that (4) and (5) hold.*

Moreover, if we suppose that $(X_i)_{i\in\mathbb{N}}$ is uniformly bounded, we obtain the result of Ibragimov [6].

Corollary 3. *If the series $\sum_{i=0}^{+\infty} \sigma_i^2$ diverges a.s. and $|X_i| \leq M < +\infty$ a.s. for all $i \geq 0$, then (4) and (5) hold.*

Example. Let $A = (A_k)_{k\in\mathbb{N}}$ be a sequence of real valued random variables such that $\sup_{k\in\mathbb{N}} \mathbb{E}(A_k^4)^{1/4} = \beta < \infty$ and consider an arbitrary sequence of variables $\zeta = (\zeta_k)_{k\in\mathbb{N}^*}$ with zero means, unit variances, bounded third moments and which are also independent of A. We definie $X = (A_{k-1}\zeta_k)_{k\in\mathbb{N}^*}$ and $\mathcal{F}_k$ the σ-algebra generated by $A_0,\, A_1, \ldots,\, A_k$.

Clearly $(X_k,\ \mathcal{F}_k,\ k \in \mathbb{N}^*)$ is a martingale difference sequence, and for all $k \in \mathbb{N}^*$,

$$\mathbb{E}(A_{k-1}^2\zeta_k^2|\mathcal{F}_{k-1}) = A_{k-1}^2 \qquad \text{a.s.},$$

$$\mathbb{E}(|A_{k-1}\zeta_k|^3\mathcal{F}_{k-1}) \leq |A_{k-1}| \sup_{i\in\mathbb{N}^*} \mathbb{E}(|\zeta_i|^3)\, A_{k-1}^2 \qquad \text{a.s.}.$$

If $(|A_k|)_{k\in\mathbb{N}}$ is nondecreasing, then using Theorem 1, one obtains

$$\sup_{x\in\mathbb{R}} \Big| F_n(x) - \phi(x) \Big| \leq c\beta \frac{\sup_{k\in\mathbb{N}^*} \mathbb{E}(|\zeta_k|^3)^{\frac{1}{4}}}{n^{\frac{1}{4}}}$$

where c is a positive constant.

3 Proof of Theorem

According to Esseen's theorem (see, e.g., ([2], 1954) p. 210 and ([8], 1955) p. 285), for all $y > 0$,

$$\sup_{x\in\mathbb{R}} \Big| F_n(x) - \phi(x) \Big| \leq \frac{1}{\pi} \int_{-y}^{y} \Big| \mathbb{E}\Big\{ \exp(\frac{itS_{\nu(n)}}{\sqrt{n}}) \Big\} - \exp(-\frac{t^2}{2}) \Big| \frac{dt}{|t|} + \frac{24}{\pi\sqrt{2\pi}y}. \tag{6}$$

Below we shall prove the following inequalities

$$\left|\mathbb{E}\left\{\exp\left(\frac{itS_{\nu(n)}}{\sqrt{n}}+\frac{t^2}{2n}\sum_{p=0}^{\nu(n)-1}\sigma_p^2\right)\right\}-1\right|\le a_n e^{\frac{t^2}{2}}\left(\frac{|t|}{3\sqrt{n}}+\frac{t^2}{4n}+\frac{a_n|t|^3}{3n^{\frac{3}{2}}}+\frac{a_n t^4}{4n^2}\right), \tag{7}$$

$$\left|\mathbb{E}\left\{\exp\left(\frac{itS_{\nu(n)}}{\sqrt{n}}+\frac{t^2}{2n}\sum_{p=0}^{\nu(n)-1}\sigma_p^2\right)\right\}-\mathbb{E}\left\{\exp\left(\frac{itS_{\nu(n)}}{\sqrt{n}}+\frac{t^2}{2}\right)\right\}\right|\le\frac{a_n t^2}{2n}\exp(\frac{t^2}{2}), \tag{8}$$

$$\left|\mathbb{E}\left\{\exp\left(\frac{itS_{\nu(n)}}{\sqrt{n}}\right)\right\}-\mathbb{E}\left\{\exp\left(\frac{itS'_{\nu(n)}}{\sqrt{n}}\right)\right\}\right|\le\frac{3a_n t^2}{2n} \tag{9}$$

where $a_n=(\mathbb{E}Y_{\nu(n)}^4)^{\frac{1}{2}}$.

3.1 Proof of the Inequality (7)

We have

$$\begin{aligned}
&\mathbb{E}\left\{\exp\left(\frac{itS_{\nu(n)}}{\sqrt{n}}+\frac{t^2}{2n}\sum_{p=0}^{\nu(n)-1}\sigma_p^2\right)\right\}-1\\
&=\sum_{k=1}^{+\infty}\mathbb{E}\left\{\left(\exp\left(\frac{itS_k}{\sqrt{n}}+\frac{t^2}{2n}\sum_{p=0}^{k-1}\sigma_p^2\right)-1\right)I_{\nu(n)=k}\right\}\\
&=\sum_{k=1}^{+\infty}\sum_{j=1}^{k}\mathbb{E}\left\{\exp\left(\frac{itS_{j-1}}{\sqrt{n}}+\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2\right)\left(e^{\frac{itX_j}{\sqrt{n}}}-e^{-\frac{t^2\sigma_{j-1}^2}{2n}}\right)I_{\nu(n)=k}\right\}.
\end{aligned}$$

For real x, put

$$e^{ix}=1+ix+\frac{(ix)^2}{2}+u(x),\quad e^{-x}=1-x+\beta(x)\frac{x^2}{2} \tag{$*$}$$

It is easily seen that, for all $x\in\mathbb{R}$

$$|u(x)|\le\frac{|x|^3}{6},\quad |u(x)|\le\frac{x^2}{2},\ \text{and}\quad |\beta(|x|)|\le 1.$$

Observing that the random variable $W_{j-1}^n=\exp\left(\frac{itS_{j-1}}{\sqrt{n}}+\frac{t^2}{2n}\sum\limits_{p=0}^{j-1}\sigma_p^2\right)$ is measurable with respect to the σ-algebra $\mathcal{F}_{j-1}$ and using the identities $(*)$, we obtain

$$\begin{aligned}
&\mathbb{E}\left\{\exp\left(\frac{itS_{\nu(n)}}{\sqrt{n}}+\frac{t^2}{2n}\sum_{p=0}^{\nu(n)-1}\sigma_p^2\right)\right\}-1\\
&=\sum_{k=1}^{+\infty}\sum_{j=1}^{k}\mathbb{E}\left\{W_{j-1}^n\mathbb{E}\left\{\left(\frac{itX_j}{\sqrt{n}}-\frac{t^2X_j^2}{2n}+u(\frac{tX_j}{\sqrt{n}})+\frac{t^2\sigma_{j-1}^2}{2n}+\beta(\frac{t^2\sigma_{j-1}^2}{2n})\frac{t^4\sigma_{j-1}^4}{8n^2}\right)I_{\nu(n)=k}|\mathcal{F}_{j-1}\right\}\right\}
\end{aligned} \tag{10}$$

Since $\{\nu(n)=k\}$ is measurable with respect to the σ-algebra $\mathcal{F}_k$, for all $j \geq 2$, we have

$$\sum_{k=1}^{j-1} \mathbb{E}\{X_j I_{\nu(n)=k} | \mathcal{F}_{j-1}\} = \sum_{k=1}^{j-1} \mathbb{E}\{(X_j^2 - \sigma_{j-1}^2) I_{\nu(n)=k} | \mathcal{F}_{j-1}\} = 0.$$

On the other hand, for all $j \geq 1$ we have

$$\sum_{k=1}^{+\infty} \mathbb{E}\{X_j I_{\nu(n)=k} | \mathcal{F}_{j-1}\} = \sum_{k=1}^{+\infty} \mathbb{E}\{(X_j^2 - \sigma_{j-1}^2) I_{\nu(n)=k} | \mathcal{F}_{j-1}\} = 0.$$

It follows that, for all $j \geq 1$

$$\sum_{k \geq j} \mathbb{E}\{X_j I_{\nu(n)=k} | \mathcal{F}_{j-1}\} = \sum_{k \geq j} \mathbb{E}\{(X_j^2 - \sigma_{j-1}^2) I_{\nu(n)=k} | \mathcal{F}_{j-1}\} = 0.$$

So, from (10) we derive

$$\begin{aligned}
&\left| \mathbb{E}\left\{ \exp\left(\frac{itS_{\nu(n)}}{\sqrt{n}} + \frac{t^2}{2n} \sum_{p=0}^{\nu(n)-1} \sigma_p^2 \right) \right\} - 1 \right| \\
&= \left| \sum_{k=1}^{+\infty} \sum_{j=1}^{k} \mathbb{E}\left\{ W_{j-1}^n \mathbb{E}\left\{ \left(u(\frac{tX_j}{\sqrt{n}}) + \beta(\frac{t^2\sigma_{j-1}^2}{2n}) \frac{t^4\sigma_{j-1}^4}{8n^2} \right) I_{\nu(n)=k} | \mathcal{F}_{j-1} \right\} \right\} \right| \\
&\leq \sum_{k=1}^{+\infty} \sum_{j=1}^{k} \mathbb{E}\left\{ \exp\left(\frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right) \mathbb{E}\left\{ \left(\frac{|t|^3 |X_j|^3}{6n^{\frac{3}{2}}} + \frac{t^4 \sigma_{j-1}^4}{8n^2} \right) I_{\nu(n)=k} | \mathcal{F}_{j-1} \right\} \right\}. \qquad (11)
\end{aligned}$$

For any $j \geq 2$ and any real function ψ such that $\mathbb{E}(\psi(X_k)) < \infty$ for any positive k, we have

$$\begin{aligned}
&\sum_{k=1}^{j-1} \mathbb{E}\left\{ \exp\left(\frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right) \mathbb{E}\left\{ \psi(X_j)\, I_{\nu(n)=k} | \mathcal{F}_{j-1} \right\} \right\} \\
&= \sum_{k=1}^{j-1} \mathbb{E}\left\{ \exp\left(\frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right) \mathbb{E}\left\{ \psi(X_j) | \mathcal{F}_{j-1} \right\} I_{\nu(n)=k} \right\}. \qquad (12)
\end{aligned}$$

On the other hand, for all $j \geq 1$, we have

$$\begin{aligned}
&\sum_{k=1}^{+\infty} \mathbb{E}\left\{ \exp\left(\frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right) \mathbb{E}\left\{ \psi(X_j)\, I_{\nu(n)=k} | \mathcal{F}_{j-1} \right\} \right\} \\
&= \mathbb{E}\left\{ \exp\left(\frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right) \psi(X_j) \right\} \\
&= \sum_{k=1}^{+\infty} \mathbb{E}\left\{ \exp\left(\frac{t^2}{2n} \sum_{p=0}^{j-1} \sigma_p^2 \right) \mathbb{E}\left\{ \psi(X_j) | \mathcal{F}_{j-1} \right\} I_{\nu(n)=k} \right\}. \qquad (13)
\end{aligned}$$

It follows from (12) and (13) that

$$\begin{aligned}
&\sum_{j=1}^{+\infty}\sum_{k\geq j}\mathbb{E}\Big\{\exp\Big(\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2\Big)\mathbb{E}\Big\{\psi(X_j)\,I_{\nu(n)=k}|\mathcal{F}_{j-1}\Big\}\Big\}\\
&=\sum_{j=1}^{+\infty}\sum_{k\geq j}\mathbb{E}\Big\{\exp\Big(\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2\Big)\mathbb{E}\Big\{\psi(X_j)|\mathcal{F}_{j-1}\Big\}\,I_{\nu(n)=k}\Big\}. \qquad (14)
\end{aligned}$$

Applying (11) and (14) for $\psi(x)=|x|^3$ we deduce that

$$\begin{aligned}
&\Big|\mathbb{E}\Big\{\exp\Big(\frac{itS_{\nu(n)}}{\sqrt{n}}+\frac{t^2}{2n}\sum_{p=0}^{\nu(n)-1}\sigma_p^2\Big)\Big\}-1\Big|\\
&\leq\sum_{k=1}^{+\infty}\sum_{j=1}^{k}\mathbb{E}\Big\{\exp\Big(\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2\Big)\Big(\mathbb{E}\Big\{\frac{|t|^3|X_j|^3}{6n^{\frac{3}{2}}}|\mathcal{F}_{j-1}\Big\}I_{\nu(n)=k}+\mathbb{E}\Big\{\frac{t^4\sigma_{j-1}^4}{8n^2}I_{\nu(n)=k}|\mathcal{F}_{j-1}\Big\}\Big)\Big\}\\
&\leq\sum_{k=1}^{+\infty}\sum_{j=1}^{k}\mathbb{E}\Big\{\exp\Big(\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2\Big)\Big(\frac{|t|^3Y_{j-1}\,\sigma_{j-1}^2}{6n^{\frac{3}{2}}}I_{\nu(n)=k}+\frac{t^4\sigma_{j-1}^4}{8n^2}I_{\nu(n)=k}\Big)\Big\} \qquad (15)
\end{aligned}$$

By the Hölder inequality, for all $j\in\mathbb{N}^*$

$$\sigma_{j-1}^2=\mathbb{E}(X_j^2|\mathcal{F}_{j-1})\leq\mathbb{E}(|X_j|^3|\mathcal{F}_{j-1})^{\frac{2}{3}}\leq Y_{j-1}^{\frac{2}{3}}\,\sigma_{j-1}^{\frac{4}{3}}\qquad a.s.,$$

whence

$$\sigma_{j-1}^2\leq Y_{j-1}^2\qquad a.s. \qquad (16)$$

From (15), (16) and using the fact that $Y_k\geq Y_{j-1}\geq 1$ for all $j\leq k$, we deduce that

$$\begin{aligned}
&\Big|\mathbb{E}\Big\{\exp\Big(\frac{itS_{\nu(n)}}{\sqrt{n}}+\frac{t^2}{2n}\sum_{p=0}^{\nu(n)-1}\sigma_p^2\Big)\Big\}-1\Big|\\
&\leq\Big(\frac{|t|^3}{6n^{\frac{3}{2}}}+\frac{t^4}{8n^2}\Big)\sum_{k=1}^{+\infty}\sum_{j=1}^{k}\mathbb{E}\Big\{Y_{j-1}^2\sigma_{j-1}^2\exp\Big(\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2\Big)I_{\nu(n)=k}\Big\}\\
&\leq\Big(\frac{|t|^3}{6n^{\frac{3}{2}}}+\frac{t^4}{8n^2}\Big)\sum_{k=1}^{+\infty}\mathbb{E}\Big\{Y_k^2\sum_{j=1}^{k}\sigma_{j-1}^2\exp\Big(\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2\Big)I_{\nu(n)=k}\Big\}. \qquad (17)
\end{aligned}$$

To bound up the terms appearing in (17), we will use the following elementary lemma.

Lemma 1. *Let $k\geq 1$, then on the event $\{\nu(n)=k\}$ we have*

$$\sum_{j=1}^{k}\exp\Big(\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2\Big)\frac{t^2}{2n}\sigma_{j-1}^2\leq\exp(\frac{t^2}{2})\Big(1+\frac{Y_k^2t^2}{n}\Big).$$

Proof of Lemma. On the event $\{\nu(n)=k\}$, we have

$$\exp(\frac{t^2}{2}) \geq \exp\left(\frac{t^2}{2n}\sum_{p=0}^{k-1}\sigma_p^2\right) - \exp\left(\frac{t^2}{2n}\sigma_0^2\right)$$
$$\geq \sum_{j=1}^{k-1}\exp\left(\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2\right)\left(\exp(\frac{t^2\sigma_j^2}{2n})-1\right)$$

Using the inequality, $\exp(x)-1\geq x$ for all $x\geq 0$, one obtains

$$\exp(\frac{t^2}{2}) \geq \sum_{j=1}^{k-1}\exp\left(\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2\right)\frac{t^2}{2n}\sigma_j^2.$$

Therefore

$$\begin{aligned}
&\sum_{j=1}^{k-1}\exp\left(\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2\right)\frac{t^2}{2n}\sigma_{j-1}^2\\
&\leq \sum_{j=1}^{k-1}\exp\left(\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2\right)\frac{t^2}{2n}(\sigma_{j-1}^2-\sigma_j^2)+\exp(\frac{t^2}{2})\\
&= \sum_{j=1}^{k-2}\left(\exp\left(\frac{t^2}{2n}\sum_{p=0}^{j}\sigma_p^2\right)-\exp\left(\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2\right)\right)\frac{t^2}{2n}\sigma_j^2-\frac{t^2}{2n}\exp\left(\frac{t^2}{2n}\sum_{p=0}^{k-2}\sigma_p^2\right)\sigma_{k-1}^2\\
&\quad+\frac{t^2}{2n}\exp(\frac{t^2}{2n}\sigma_0^2)\sigma_0^2+\exp(\frac{t^2}{2})\\
&\leq \frac{t^2}{2n}Y_k^2\sum_{j=1}^{k-2}\left(\exp\left(\frac{t^2}{2n}\sum_{p=0}^{j}\sigma_p^2\right)-\exp\left(\frac{t^2}{2n}\sum_{p=0}^{j-1}\sigma_p^2\right)\right)+\frac{t^2}{2n}Y_k^2\exp(\frac{t^2}{2n}\sigma_0^2)+\exp(\frac{t^2}{2})\\
&\leq \left(1+\frac{t^2}{2n}Y_k^2\right)\exp(\frac{t^2}{2}).
\end{aligned}$$

We conclude the proof of the lemma by noting that $\sigma_{k-1}^2\leq Y_k^2$ and $\sum_{p=0}^{k-1}\sigma_p^2\leq n$ a.s..

Finally, according to Lemma 1 and the (17) we get

$$\left|\mathbb{E}\left\{\exp\left(\frac{itS_{\nu(n)}}{\sqrt{n}}+\frac{t^2}{2n}\sum_{p=0}^{\nu(n)-1}\sigma_p^2\right)\right\}-1\right|\leq a_n\exp(\frac{t^2}{2})\left(\frac{|t|}{3\sqrt{n}}+\frac{t^2}{4n}+\frac{a_n|t|^3}{3n^{\frac{3}{2}}}+\frac{a_nt^4}{4n^2}\right),$$

where $a_n=(\mathbb{E}Y_{\nu(n)}^4)^{\frac{1}{2}}$.

3.2 Proof of the Inequality (8)

Using (1) and the inequality $|1-\exp(-x)| \le x$, for all $x \ge 0$ we see that

$$\begin{aligned}
&\Big|\mathbb{E}\Big\{\exp\Big(\frac{itS_{\nu(n)}}{\sqrt{n}} + \frac{t^2}{2n}\sum_{p=0}^{\nu(n)-1}\sigma_p^2\Big)\Big\} - \mathbb{E}\Big\{\exp\Big(\frac{itS_{\nu(n)}}{\sqrt{n}} + \frac{t^2}{2}\Big)\Big\}\Big| \\
&= \Big|\mathbb{E}\Big\{\exp\Big(\frac{itS_{\nu(n)}}{\sqrt{n}} + \frac{t^2}{2}\Big)\Big(\exp\Big(-\frac{t^2}{2n}\gamma(n)\sigma^2_{\nu(n)}\Big) - 1\Big)\Big\}\Big| \\
&\le \mathbb{E}\Big\{\Big|1-\exp\Big(-\frac{t^2}{2n}\gamma(n)\sigma^2_{\nu(n)}\Big)\Big|\Big\}\exp(\frac{t^2}{2}) \\
&\le \mathbb{E}\Big\{\frac{t^2}{2n}|\gamma(n)|\sigma^2_{\nu(n)}\Big\}\exp(\frac{t^2}{2}) \\
&\le (\mathbb{E}Y^4_{\nu(n)})^{\frac{1}{2}}\frac{t^2}{2n}\exp(\frac{t^2}{2}).
\end{aligned}$$

Therefore (8) holds true.

From (7) and (8) we conclude that

$$\Big|\mathbb{E}\Big\{\exp(\frac{itS_{\nu(n)}}{\sqrt{n}})\Big\} - \exp(-\frac{t^2}{2})\Big| \le a_n\Big(\frac{|t|}{3\sqrt{n}} + \frac{3t^2}{4n} + \frac{|t|^3}{3n^{\frac{3}{2}}}a_n + \frac{t^4}{4n^2}a_n\Big).$$

Using Esseen's theorem, we derive

$$\sup_{x\in\mathbb{R}}\Big|F_n(x)-\phi(x)\Big| \le \frac{a_n}{\pi}\int_{-y}^{y}\Big(\frac{1}{3\sqrt{n}} + \frac{3|t|}{4n} + \frac{t^2}{3n^{\frac{3}{2}}}a_n + \frac{|t|^3}{4n^2}a_n\Big)dt + \frac{24}{\pi\sqrt{2\pi}y}.$$

Hence

$$\sup_{x\in\mathbb{R}}\Big|F_n(x)-\phi(x)\Big| \le \frac{a_n}{\pi}\Big(\frac{2y}{3\sqrt{n}} + \frac{3y^2}{4n} + \frac{2y^3}{9n^{\frac{3}{2}}}a_n + \frac{y^4}{8n^2}a_n\Big) + \frac{24}{\pi\sqrt{2\pi}y}.$$

Choosing y in such a way that $y/\sqrt{n} = 1/(ya_n)$, i.e. $y = (n/a_n^2)^{\frac{1}{4}}$, we infer that

$$\sup_{x\in\mathbb{R}}\Big|F_n(x)-\phi(x)\Big| \le \frac{{a_n}^{\frac{1}{2}}}{\pi n^{\frac{1}{4}}}\Big(11 + \frac{3}{4n^{\frac{1}{4}}} + \frac{2}{9n^{\frac{1}{2}}} + \frac{1}{8n^{\frac{3}{4}}}\Big).$$

The proof of the inequality (2) in theorem is complete.

3.3 Proof of the Inequality (9)

Observing that the random events $\{\gamma(n)\le x\}\cap\{\nu(n)=k\}$ and consequently the random variables $\sqrt{\gamma(n)}I_{\nu(n)=k}$ are measurable with respect to $\mathcal{F}_k$, we find that

$$\begin{aligned}
&\Big|\mathbb{E}\Big\{\exp\Big(\frac{itS_{\nu(n)}}{\sqrt{n}}\Big)\Big\} - \mathbb{E}\Big\{\exp\Big(\frac{itS'_{\nu(n)}}{\sqrt{n}}\Big)\Big\}\Big| \\
&= \Big|\sum_{k=0}^{+\infty}\mathbb{E}\Big\{\Big(\exp\Big(\frac{itS_k}{\sqrt{n}}\Big) - \exp\Big(\frac{itS_k}{\sqrt{n}} + \frac{it\sqrt{\gamma(n)}}{\sqrt{n}}x_{\nu(n)+1}\Big)\Big)I_{\nu(n)=k}\Big\}\Big|
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{k=0}^{+\infty}\Big|\mathbb{E}\Big\{\exp\Big(\frac{itS_k}{\sqrt{n}}\Big)\Big(1-\exp\Big(\frac{it\sqrt{\gamma(n)}}{\sqrt{n}}X_{\nu(n)+1}\Big)\Big)I_{\nu(n)=k}\Big\}\Big|\\
&= \sum_{k=0}^{+\infty}\Big|\mathbb{E}\Big\{\exp\Big(\frac{itS_k}{\sqrt{n}}\Big)\Big(-\frac{it\sqrt{\gamma(n)}}{\sqrt{n}}X_{k+1}+\frac{t^2}{2n}\gamma(n)X_{k+1}^2-u(\frac{t\sqrt{\gamma(n)}}{\sqrt{n}}X_{k+1})\Big)I_{\nu(n)=k}\Big\}\Big|\\
&= \sum_{k=0}^{+\infty}\Big|\mathbb{E}\Big\{\exp\Big(\frac{itS_k}{\sqrt{n}}\Big)\mathbb{E}\Big\{-\frac{it\sqrt{\gamma(n)}}{\sqrt{n}}X_{k+1}+\frac{t^2}{2n}\gamma(n)X_{k+1}^2-u(\frac{t\sqrt{\gamma(n)}X_{k+1}}{\sqrt{n}})|\mathcal{F}_k\Big\}I_{\nu(n)=k}\Big\}\Big|\\
&= \sum_{k=0}^{+\infty}\Big|\mathbb{E}\Big\{\exp\Big(\frac{itS_k}{\sqrt{n}}\Big)\Big(\frac{t^2}{2n}\gamma(n)X_{k+1}^2-u(\frac{t}{\sqrt{n}}\sqrt{\gamma(n)}\,X_{k+1})\Big)I_{\nu(n)=k}\Big\}\Big|\\
&\leq \sum_{k=0}^{+\infty}\mathbb{E}\Big\{I_{\nu(n)=k}\,\frac{3t^2}{2n}\gamma(n)X_{k+1}^2\Big\}\\
&\leq \frac{3t^2}{2n}\sum_{k=0}^{+\infty}\mathbb{E}\Big\{I_{\nu(n)=k}\ \mathbb{E}\{X_{k+1}^2|\mathcal{F}_k\}\Big\}\\
&\leq \frac{3t^2}{2n}\,\mathbb{E}(Y_{\nu(n)}^4)^{\frac{1}{2}}.
\end{aligned}
$$

The proof of the inequality (9) is complete.

3.4 Proof of the inequality (3).

According to Esseen's theorem where $y=(n/a_n^2)^{\frac{1}{4}}$ and the inequality (9), one obtains

$$
\begin{aligned}
\sup_{x\in\mathbb{R}}\Big|H_n(x)-\phi(x)\Big| &\leq \frac{1}{\pi}\int_{-y}^{y}\Big|\mathbb{E}\Big\{\exp(\frac{itS'_{\nu(n)}}{\sqrt{n}})\Big\}-\exp(-\frac{t^2}{2})\Big|\,\frac{dt}{|t|}+\frac{24}{\pi\sqrt{2\pi}y}\\
&\leq \frac{a_n^{\frac{1}{2}}}{\pi n^{\frac{1}{4}}}\Big(11+\frac{3}{4n^{\frac{1}{4}}}+\frac{2}{9n^{\frac{1}{2}}}+\frac{1}{8n^{\frac{3}{4}}}\Big)+\frac{1}{\pi}\int_{-y}^{y}\frac{3|t|}{2n}E(Y_{\nu(n)}^4)^{\frac{1}{2}}\,dt\\
&\leq \frac{a_n^{\frac{1}{2}}}{\pi n^{\frac{1}{4}}}\Big(11+\frac{3}{4n^{\frac{1}{4}}}+\frac{2}{9n^{\frac{1}{2}}}+\frac{1}{8n^{\frac{3}{4}}}\Big)+\frac{3}{2\pi\sqrt{n}}\\
&\leq \frac{a_n^{\frac{1}{2}}}{\pi n^{\frac{1}{4}}}\Big(11+\frac{9}{4n^{\frac{1}{4}}}+\frac{2}{9n^{\frac{1}{2}}}+\frac{1}{8n^{\frac{3}{4}}}\Big).
\end{aligned}
$$

The proof of theorem is complete. □

Proofs of corollaries 1, 2 and 3 are easy so, it is left to the reader.

Acknowledgements: The author thanks the referee for careful reading of the manuscript and for valuable suggestions which improved the presentation of this paper.

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